

# LOGARITHMIC VECTOR FIELDS FOR QUASIHOMOGENEOUS CURVE CONFIGURATIONS IN $\mathbb{P}^2$

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**ABSTRACT.** Let  $\mathcal{A} = \bigcup_{i=1}^r C_i \subseteq \mathbb{P}_{\mathbb{C}}^2$  be a collection of smooth plane curves, such that each singular point is quasihomogeneous. We prove that if  $C$  is a smooth curve such that each singular point of  $\mathcal{A} \cup C$  is also quasihomogeneous, then there is an elementary modification of rank two bundles, which relates the  $\mathcal{O}_{\mathbb{P}^2}$ -module  $\text{Der}(\log \mathcal{A})$  of vector fields on  $\mathbb{P}^2$  tangent to  $\mathcal{A}$  to the module  $\text{Der}(\log \mathcal{A} \cup C)$ . This yields an inductive tool for studying the splitting of the bundles  $\text{Der}(\log \mathcal{A})$  and  $\text{Der}(\log \mathcal{A} \cup C)$ , depending on the geometry of the divisor  $\mathcal{A}|_C$  on  $C$ .

## 1. INTRODUCTION

For a divisor  $Y$  in a complex manifold  $X$ , Saito [10] introduced the sheaves of logarithmic vector fields and logarithmic one forms with pole along  $Y$ :

**Definition 1.1.** *The module of logarithmic vector fields is the sheaf of  $\mathcal{O}_X$ -modules*

$$\text{Der}(\log Y)_p = \{\theta \in \text{Der}_{\mathbb{C}}(X) \mid \theta(f) \in \langle f \rangle\},$$

where  $f \in \mathcal{O}_{X,p}$  is a local defining equation for  $Y$  at  $p$ .

Saito's work generalized earlier work of Deligne [4], where the situation was studied for  $Y$  a normal crossing divisor. If  $\{x_1, \dots, x_d\}$  are local coordinates at a point  $p \in X$  and  $Y$  has local equation  $f$ , then  $\text{Der}(\log Y)_p$  is the kernel of the evaluation map  $\theta \mapsto \theta(f) \in \mathcal{O}_{Y,p}$ , so there is a short exact sequence

$$0 \longrightarrow \text{Der}(\log Y) \longrightarrow \mathcal{T}_X \longrightarrow J_Y(Y) \longrightarrow 0,$$

where  $J_Y$  is the Jacobian scheme, defined locally at  $p$  by  $\{\partial f / \partial x_1, \dots, \partial f / \partial x_d\}$ . Saito shows that  $\text{Der}(\log Y)_p$  is a free  $\mathcal{O}_{X,p}$  module iff there exist  $d$  elements

$$\theta_i = \sum_{j=1}^d f_{ij} \frac{\partial}{\partial x_j} \in \text{Der}(\log Y)_p$$

such that the determinant of the matrix  $[f_{ij}]$  is a nonzero constant multiple of the local defining equation for  $Y$ ; this is basically a consequence of the Hilbert-Burch theorem. A much studied version of this construction occurs when  $Y = V(F) \subseteq \mathbb{P}^d$  is a reduced hypersurface;  $V(F)$  may also be studied as a hypersurface in  $\mathbb{C}^{d+1}$ . In particular, if  $X = \mathbb{C}^{d+1}$  and  $F \in S = \mathbb{C}[x_0, \dots, x_d]$  is homogeneous, we write

**Definition 1.2.**  $D(V(F)) = \{\theta \in \text{Der}_{\mathbb{C}}(S) \mid \theta(F) \in \langle F \rangle\}$ .

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Since  $F$  is homogeneous,  $D(V(F))$  is a graded  $S$ -module, hence gives rise to an associated sheaf on  $\mathbb{P}^d$ . The kernel of the evaluation map  $\theta \mapsto \theta(F)$

$$D(V(F)) \rightarrow S$$

consists of the syzygies on the Jacobian ideal of  $F$ , which we denote  $D_0(V(F))$ . Since the Euler vector field  $\sum x_i \partial / \partial x_i$  gives a surjection

$$D(V(F)) \rightarrow \langle F \rangle \rightarrow 0,$$

we can split the map, hence

$$D(V(F)) \simeq D_0(V(F)) \oplus S(-1).$$

In particular, if  $F$  has degree  $n$ , then there is an exact sequence

$$0 \longrightarrow D_0(V(F)) \longrightarrow S^{d+1} \xrightarrow{\theta \mapsto \theta(F)} S(n-1) \longrightarrow S(n-1)/J_F \longrightarrow 0.$$

This shows that  $D(V(F))$  and the associated sheaf are second syzygies, so when  $d = 2$ ,  $D(V(F))$  is a vector bundle on  $\mathbb{P}^2$ . Since the depth of  $D(V(F))$  is at least two,  $D(V(F))$  is  $\Gamma_*$  of the associated sheaf; we use  $D(V(F))$  to denote both the  $S$ -module and associated sheaf. Tensoring  $D(V(F))$  with  $\mathcal{O}_{\mathbb{P}^d}(1)$  yields the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^d} & \longrightarrow & \mathcal{O}_{\mathbb{P}^d} & \longrightarrow & 0 \\ & & \downarrow \begin{bmatrix} x_0 \\ \vdots \\ x_d \end{bmatrix} & & \downarrow \begin{bmatrix} x_0 \\ \vdots \\ x_d \end{bmatrix} & & \downarrow \\ 0 & \longrightarrow & D(V(F))(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^d}^{d+1}(1) & \xrightarrow{\gamma} & \mathcal{O}_{V(F)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D(V(F))/E(1) & \longrightarrow & \mathcal{T}_{\mathbb{P}^d} & \longrightarrow & \mathcal{O}_{V(F)} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0. \end{array}$$

The map  $\gamma$  sends  $\theta = \sum f_i \partial / \partial x_i$  to  $\theta(F)$ , so  $\gamma$  surjects onto  $J_F(n)$ . Hence,

$$(1) \quad \text{Der}(\log V(F)) \simeq D(V(F))/E(1) \simeq D_0(V(F))(1).$$

In the case of generic hyperplane arrangements this is noted in [8], we include it here to make precise the relationship. A major impetus in studying  $D(V(F))$  comes from the setting of hyperplane arrangements. If  $F$  is a product of distinct linear forms, then write  $V(F) = \mathcal{A}$ . Terano's theorem [14] shows that in this setting, if  $D(\mathcal{A})$  is a free  $S$ -module, with  $D(\mathcal{A}) \simeq \oplus S(-a_i)$ , then

$$\sum h^i(\mathbb{C}^{d+1} \setminus \mathcal{A}, \mathbb{Q}) t^i = \prod (1 + a_i t).$$

**1.1. Addition-Deletion theorems.** A central tool in the study of hyperplane arrangements is an inductive method due to Terao. For a hyperplane arrangement  $\mathcal{A}$  and choice of  $H \in \mathcal{A}$ , set

$$\mathcal{A}' = \mathcal{A} \setminus H \text{ and } \mathcal{A}'' = \mathcal{A}|_H.$$

The collection  $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$  is called a *triple*, and yields a left exact sequence

$$0 \longrightarrow D(\mathcal{A}')(-1) \xrightarrow{\cdot H} D(\mathcal{A}) \longrightarrow D(\mathcal{A}'').$$

Freeness of a triple is related via:

**Theorem 1.3.** [Terao, [15]] *Let  $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$  be a triple. Then any two of the following imply the third*

- $D(\mathcal{A}) \simeq \oplus_{i=1}^n S(-b_i)$
- $D(\mathcal{A}') \simeq S(-b_n + 1) \oplus_{i=1}^{n-1} S(-b_i)$
- $D(\mathcal{A}'') \simeq \oplus_{i=1}^{n-1} S(-b_i)$

For a triple with  $\mathcal{A} \subseteq \mathbb{P}^2$ , [12] shows that after pruning the Euler derivations and sheafifying, there is an exact sequence

$$0 \longrightarrow \mathcal{D}_0'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_* \mathcal{D}_0'' \longrightarrow 0,$$

where  $i : H \hookrightarrow \mathbb{P}^2$ ;  $i_* \mathcal{D}_0'' \simeq \mathcal{O}_H(1 - |\mathcal{A}''|)$ . In [13], a version of this exact sequence (and associated addition-deletion theorem) is shown to hold for arrangements of lines and conics in  $\mathbb{P}^2$  having all singularities quasihomogeneous. In related work, Dimca-Sticlaru study the Milnor algebra of nodal curves in [5].

**1.2. Statement of results.** This paper is motivated by the following example:

**Example 1.4.** The braid arrangement  $A_3$  is depicted below;  $D(A_3)$  is free and is

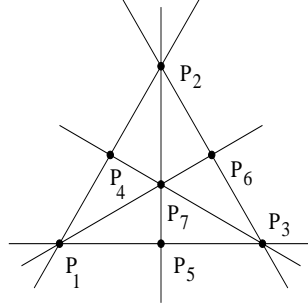


FIGURE 1. The  $A_3$ -arrangement

isomorphic to  $S(-1) \oplus S(-2) \oplus S(-3)$ . There is a three dimensional family of cubics passing through the seven singular points of  $A_3$ . A generic member  $C$  of this family is smooth, and  $D(A_3 \cup C) \simeq S(-1) \oplus S(-4) \oplus S(-4)$ . Our main result is Theorem 1.5, which combined with Proposition 1.6 explains this example.

**Theorem 1.5.** *Let  $\mathcal{A} = \bigcup_{i=1}^r C_i$  and  $\mathcal{A} \cup C$  be collections of smooth curves in  $\mathbb{P}^2$ , such that all singular points of  $\mathcal{A}$  and  $\mathcal{A} \cup C$  are quasihomogeneous. Then*

$$0 \longrightarrow \text{Der}(\log \mathcal{A})(-n) \longrightarrow \text{Der}(\log \mathcal{A} \cup C) \longrightarrow \mathcal{O}_C(-K_C - R) \longrightarrow 0$$

*is exact, where  $R = (A \cap C)_{\text{red}}$  is the reduced scheme of  $C \cap \mathcal{A}$  and  $\deg(C) = n$ .*

We prove the theorem in §3. For addition-deletion arguments, we will need

**Proposition 1.6.** *Suppose  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  is an exact sequence of graded  $S = k[x_0, \dots, x_n]$ -modules, with  $A$  and  $B$  of rank two and projective dimension at most one. Then any two of the following imply the third*

- (1)  $A$  is free with generators in degrees  $\{a, b\}$ .
- (2)  $B$  is free with generators in degrees  $\{c, d\}$ .
- (3)  $C$  has Hilbert series  $\frac{t^c + t^d - t^a - t^b}{(1-t)^{n+1}}$ .

*Proof.* That (1) and (2) imply (3) is trivial. If (1) and (3) hold and  $B$  is not free, then  $\text{pdim}(B) = 1$ , so  $B$  has a minimal free resolution of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0.$$

Note that  $F_1 = \oplus S(-e_i)$  and  $F_0 = \oplus S(-e_i) \oplus S(-c) \oplus S(-d)$ , since by additivity the Hilbert series of  $B$  is  $\frac{t^c + t^d}{(1-t)^{n+1}}$ . Without loss of generality suppose  $d \geq c$ . If the largest  $e_i > d$ , then since the resolution is minimal, no term of  $F_1$  can map to the generator in degree  $e_i$ , which forces  $B$  to have a free summand of rank one. Since  $B$  has rank two this forces  $B$  to be free. Next suppose the largest  $e_i = d$ . Since the Hilbert series of  $B$  is  $\frac{t^c + t^d}{(1-t)^{n+1}}$ ,  $B$  has a minimal generator of degree  $d$ . But then no element of  $F_1$  can be a relation involving that generator, and again  $B$  has a free summand. This obviously also works when the largest  $e_i$  is less than  $d$ , and shows that (1) and (3) imply (2). The argument that (2) and (3) imply (1) is similar.  $\square$

If a bundle of the form  $\text{Der}(\log \mathcal{A})$  splits as a sum of line bundles (is free), then applying  $\Gamma_*$  to the short exact sequence of Theorem 1.5 yields a short exact sequence of modules, since  $H^1(\oplus \mathcal{O}_{\mathbb{P}^2}(a_i)) = 0$ . Then by Proposition 1.6, on  $\mathbb{P}^2$  the freeness of  $\text{Der}(\log \mathcal{A} \cup C)$  follows if appropriate numerical conditions hold. In contrast to arrangements of rational curves (where the Hilbert series of  $\Gamma_*(\mathcal{O}_C(-K_C - R))$  depends only on the degree of  $R$ , since  $C \simeq \mathbb{P}^1$ ), for curves of positive genus the Hilbert series of  $\Gamma_*(\mathcal{O}_C(-K_C - R))$  depends on subtle geometry.

## 2. QUASIHOMOGENEOUS PLANE CURVES

Let  $C = V(Q)$  be a reduced (but not necessarily irreducible) curve in  $\mathbb{C}^2$ , let  $(0, 0) \in C$ , and let  $\mathbb{C}\{x, y\}$  denote the ring of convergent power series.

**Definition 2.1.** *The Milnor number of  $C$  at  $(0, 0)$  is*

$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

To define  $\mu_p$  for an arbitrary point  $p$ , we translate so that  $p$  is the origin.

**Definition 2.2.** *The Tjurina number of  $C$  at  $(0, 0)$  is*

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle.$$

**Definition 2.3.** *A singularity is quasihomogeneous iff there exists a holomorphic change of variables so the defining equation becomes weighted homogeneous;  $f(x, y) = \sum c_{ij} x^i y^j$  is weighted homogeneous if there exist rational numbers  $\alpha, \beta$  such that  $\sum c_{ij} x^{i \cdot \alpha} y^{j \cdot \beta}$  is homogeneous.*

In [11], Saito shows that if  $f$  is a convergent power series with isolated singularity at the origin, then  $f$  is in the ideal generated by the partial derivatives if and only if  $f$  is quasihomogeneous. As noted in §1.3 of [13], if all the singular points of  $V(Q) \subseteq \mathbb{P}^2$  are quasihomogeneous, then

$$\deg(J_Q) = \sum_{p \in \text{Sing}(V(Q))} \mu_p(Q).$$

**Lemma 2.4.** [[17], Theorem 6.5.1] *Let  $X$  and  $Y$  be two reduced plane curves with no common component, meeting at a point  $p$ . Then*

$$\mu_p(X \cup Y) = \mu_p(X) + \mu_p(Y) + 2(X \cdot Y)_p - 1,$$

where  $(X \cdot Y)_p$  is the intersection number of  $X$  and  $Y$  at  $p$ .

**Proposition 2.5.** *Let  $\mathcal{A}$  and  $\mathcal{A} \cup C$  be quasihomogeneous, with  $C = V(f)$  of degree  $n$  and  $\mathcal{A} = V(Q)$  with  $Q$  of degree  $m$ . Then the Hilbert polynomial  $\text{HP}(\text{coker}(f), t)$  of the cokernel of the multiplication map*

$$0 \longrightarrow D(\mathcal{A})(-n)/E \xrightarrow{\cdot f} D(\mathcal{A} \cup C)/E$$

is  $nt + 3(1 - g_C) - n - k$ , where  $k = |C \cap \mathcal{A}|$ .

*Proof.* By Equation 1 and the exact sequences

$$0 \longrightarrow D_0(\mathcal{A} \cup C) \longrightarrow S^3 \longrightarrow S(m+n-1) \longrightarrow S(m+n-1)/J_{\mathcal{A} \cup C} \longrightarrow 0,$$

$$0 \longrightarrow D_0(\mathcal{A})(-n) \longrightarrow S^3(-n) \longrightarrow S(m-n-1) \longrightarrow S(m-n-1)/J_{\mathcal{A}} \longrightarrow 0,$$

it follows that

$$\text{HP}(D_0(\mathcal{A} \cup C), t) = 3 \binom{t+2}{2} - \binom{t+1+m+n}{2} + \deg(J_{\mathcal{A} \cup C})$$

$$\text{HP}(D_0(\mathcal{A})(-n), t) = 3 \binom{t+2-n}{2} - \binom{t+1+m-n}{2} + \deg(J_{\mathcal{A}}),$$

so that  $\text{HP}(D_0(\mathcal{A} \cup C), t) - \text{HP}(D_0(\mathcal{A})(-n), t)$  is equal to

$$\deg(J_{\mathcal{A} \cup C}) - \deg(J_{\mathcal{A}}) + nt - n(2m+1) + \frac{3}{2}(3-n)(n).$$

Since  $C$  is smooth of degree  $n$ ,  $\frac{3}{2}(3-n)(n) = 3(1 - g_C)$ . To compute  $\deg(J_{\mathcal{A} \cup C}) - \deg(J_{\mathcal{A}})$ , note that since all singularities of  $\mathcal{A}$  and  $\mathcal{A} \cup C$  are quasihomogeneous,

$$\deg(J_{\mathcal{A} \cup C}) = \sum_{p \in \text{Sing}(\mathcal{A} \cup C)} \mu_p(\mathcal{A} \cup C) \text{ and } \deg(J_{\mathcal{A}}) = \sum_{p \in \text{Sing}(\mathcal{A})} \mu_p(\mathcal{A}).$$

Let  $\alpha$  be the sum of Milnor numbers of points off  $C$ , so

$$\deg(J_{\mathcal{A} \cup C}) = \alpha + \sum_{p \in C \cap \mathcal{A}} \mu_p(\mathcal{A} \cup C).$$

Since  $\mu_p(C) = 0$ , by Lemma 2.4, the previous quantity equals

$$\alpha + \sum_{p \in C \cap \mathcal{A}} (\mu_p(\mathcal{A}) + 2(C \cdot \mathcal{A})_p - 1).$$

As  $\deg(J_{\mathcal{A}}) = \alpha + \sum_{p \in C \cap \mathcal{A}} \mu_p(\mathcal{A})$  and  $|C \cap \mathcal{A}| = k$ , we obtain:

$$\deg(J_{\mathcal{A} \cup C}) - \deg(J_{\mathcal{A}}) = 2 \sum_{p \in C \cap \mathcal{A}} (C \cdot \mathcal{A})_p - k.$$

By Bezout's theorem,

$$\sum_{p \in C \cap A} (C \cdot \mathcal{A}) = mn, \text{ so } \deg(J_{A \cup C}) - \deg(J_A) = 2mn - k,$$

hence the Hilbert polynomial of the cokernel is

$$nt - n(2m + 1) + 3(1 - g_C) + 2mn - k = nt - n - k + 3(1 - g_C).$$

□

### 3. MAIN THEOREM

We now prove Theorem 1.5. First we show that the sheaf associated to the cokernel of the multiplication map

$$0 \longrightarrow \text{Der}(\log A)(-n) \xrightarrow{\cdot f} \text{Der}(\log A \cup C)$$

is isomorphic to  $\mathcal{O}_C(D)$ , where  $D$  is a divisor of degree  $3n - n^2 - k$ . Consider the commuting diagram below

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Der}(\log A)(-n) & \longrightarrow & \mathcal{T}_{\mathbb{P}^d}(-n) & \longrightarrow & J_A(-n) \longrightarrow 0 \\ & & \downarrow \cdot f & & \downarrow \cdot f & & \downarrow \cdot f \\ 0 & \longrightarrow & \text{Der}(\log A \cup C) & \longrightarrow & \mathcal{T}_{\mathbb{P}^d} & \longrightarrow & J_{A \cup C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{coker}(f) & \longrightarrow & \mathcal{T}_{\mathbb{P}^d}/f \cdot \mathcal{T}_{\mathbb{P}^d} & \longrightarrow & J_{A \cup C}/J_A(-n) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Exactness of the top two rows follows from the definition, and since all the modules are torsion free, multiplication by  $f$  gives an inclusion. Exactness of the bottom row then follows from the snake lemma. The exact sequence

$$0 \longrightarrow S \longrightarrow S^3(1) \longrightarrow \Gamma_*(\mathcal{T}_{\mathbb{P}^2}) \longrightarrow 0,$$

shows that the Hilbert polynomial of  $\Gamma_*(\mathcal{T}_{\mathbb{P}^2}/f \cdot \mathcal{T}_{\mathbb{P}^2})$  is  $2nt + 6n - n^2$ . Since

$$\mathcal{T}_{\mathbb{P}^2}/f \cdot \mathcal{T}_{\mathbb{P}^2} \simeq \mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C,$$

and  $\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C$  is a locally free rank two  $\mathcal{O}_C$ -module,  $\text{coker}(f)$  is a torsion free submodule of  $\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C$ , hence locally free on  $C$ , of rank one or two. Since

$$\text{Der}(\log \mathcal{A} \cup C) \simeq D_0(\mathcal{A} \cup C)(1),$$

by Proposition 2.5, the Hilbert polynomial of  $\Gamma_*(\text{coker}(f))$  is  $nt - k + 3(1 - g_C)$ , so  $\text{coker}(f) \simeq \mathcal{O}_C(D)$ . To determine the degree of  $D$ , we compute

$$\begin{aligned} \text{HP}(\Gamma_*(\text{coker}(f)), t) &= h^0(\mathcal{O}_C(D + tH)), \quad t \gg 0 \\ &= \deg(D + tH) + 1 - g_C \\ &= \deg(D) + nt + 1 - g_C. \end{aligned}$$

Equating this with the previous expression shows that  $\deg(D) = 2 - 2g_C - k$ . By adjunction,

$$2g_C - 2 = C(C + K_{\mathbb{P}^2}) = nH(nH - 3H) = n^2 - 3n,$$

so  $\deg(D) = 3n - n^2 - k$ . Notice this shows the left hand column is an elementary modification of bundles. To conclude, consider the short exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C \longrightarrow N_{C/\mathbb{P}^2} \longrightarrow 0.$$

Since  $\mathcal{O}_C(D)$  comes from the restriction of  $\text{Der}(\log \mathcal{A} \cup C)$  to  $C$ , it must actually be a subbundle of  $\mathcal{T}_C$ , which by adjunction is isomorphic to  $\mathcal{O}_C((3-n)H)$ . For the same reason, sections must vanish at points of  $\mathcal{A} \cap C$ , so that  $\mathcal{O}_C(D) \subseteq \mathcal{O}_C((3-n)H - R)$ . But the degree of this last bundle is  $3n - n^2 - k$ , so we have equality, which concludes the proof.  $\square$

**3.1. Castelnuovo-Mumford regularity.** Theorem 1.5 yields bounds on the Castelnuovo-Mumford regularity of logarithmic vector bundles and, by dualizing, for logarithmic one forms for quasihomogeneous curve arrangements.

**Definition 3.1.** A coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^d$  is  $j$ -regular iff  $H^i \mathcal{F}(j-i) = 0$  for every  $i \geq 1$ . The smallest number  $j$  such that  $\mathcal{F}$  is  $j$ -regular is  $\text{reg}(\mathcal{F})$ .

**Lemma 3.2.** With the hypotheses of Theorem 1.5,

$$\text{reg}(\mathcal{D}_0(\mathcal{A} \cup C)) \leq \max\{\text{reg}(\mathcal{D}_0(\mathcal{A})) + n, 2n - 4 + \frac{k}{n}\}.$$

*Proof.* The short exact sequence

$$0 \longrightarrow D_0(\mathcal{A})(-n) \longrightarrow D_0(\mathcal{A} \cup C) \longrightarrow \mathcal{O}_C(-K_C - H - R) \longrightarrow 0$$

gives a long exact sequence in cohomology, so if  $D_0(\mathcal{A})$  is  $a$ -regular, then

$$h^1(D_0(\mathcal{A})(a-1)) = 0 = h^2(D_0(\mathcal{A})(a-2)).$$

So if  $t - n - 1 \geq a - 1$  and  $t - n - 2 \geq a - 2$  we have that

$$h^1(D_0(\mathcal{A})(t-n-1)) = 0 = h^2(D_0(\mathcal{A})(t-n-2)).$$

This gives vanishings if  $t - n \geq a$ , that is, if  $t \geq \text{reg } D_0(\mathcal{A}) + n$ . The result will follow if

$$h^1 \mathcal{O}_C((t-2)H - K_C - R) = h^0 \mathcal{O}_C((2-t)H + 2K_C + R) = 0.$$

Using that  $K_C = (n-3)H$ , this holds if  $\deg((t-2)H + 2K_C + R) < 0$ , that is, when

$$t > 2n - 4 + \frac{k}{n}$$

The result follows.  $\square$

The previous proof shows that the Hilbert function of  $\Gamma_* \mathcal{O}_C(-K_C - H - R)$  is equal to the Hilbert polynomial when  $t > 2n - 4 + \frac{k}{n}$ .

**Proposition 3.3.** With the hypotheses of Theorem 1.5, the Hilbert function of  $\Gamma_* \mathcal{O}_C(-K_C - H - R)$  is  $nt + 3(1 - g_C) - n - k$  for

$$t < n - 2 + \frac{k}{n} \text{ or } t > 2n - 4 + \frac{k}{n}.$$

*Proof.* If  $t < n - 2 + \frac{k}{n}$ , then the degree of  $(t-1)H - K_C - R$  is negative, so there can be no sections.  $\square$

## 4. EXAMPLES

**Example 4.1.** We analyze Example 1.4 in more detail. Since  $C$  is a cubic curve,  $g_C = 1$  and  $K_C \simeq \mathcal{O}_C$ . Since  $C$  meets every line of  $\mathcal{A}$  in three points,  $k = |C \cap \mathcal{A}| = 7$ . By Proposition 3.3, the Hilbert polynomial and Hilbert function of  $\Gamma_* \mathcal{O}_C(-K_C - H - R)$  agree for  $t < \frac{10}{3}$  and  $t > \frac{13}{3}$ , and so applying Proposition 2.5 and Theorem 1.5 we have

$t$	0	1	2	3	4	5	6	7
$h^0((t-1)H - K_C - R)$	0	0	0	0	?	5	8	11

Now,  $H^0((4-1)H - R)$  consists of cubics through the seven singular points of  $\mathcal{A}$ , and this space has dimension 2, since  $C$  is itself one of the three cubics, so is not counted. Thus, in this example the Hilbert polynomial  $3t - 10$  agrees with the Hilbert function for all  $t$ , and

$$\mathrm{HS}(\Gamma_* \mathcal{O}_C(-K_C - H - R), t) = \frac{2t^4 + t^5}{(1-t)^2} = \frac{2t^4 - t^5 - t^6}{(1-t)^3}.$$

Terao's result [16] on reflection arrangements shows that  $D(A_3) \simeq S(-1) \oplus S(-2) \oplus S(-3)$ , so

$$\mathrm{Der}(\log A_3(-3)) \simeq S(-5) \oplus S(-6).$$

Taking global sections in Theorem 1.5 and applying Proposition 1.6, we find that  $\mathrm{Der}(\log A_3 \cup C)$  is free, with

$$\begin{aligned} \mathrm{HS}(\mathrm{Der}(\log A_3 \cup C)) &= \frac{t^5 + t^6}{(1-t)^3} + \frac{2t^4 - t^5 - t^6}{(1-t)^3} \\ &= \frac{2t^4}{(1-t)^3}. \end{aligned}$$

**Example 4.2.** The reflection arrangement  $B_3$  consists of the nine planes of symmetry of a cube in  $\mathbb{R}^3$ . The intersection of  $B_3$  with the affine chart  $U_z$  is depicted below (this does not show the line at infinity  $z = 0$ ). By [16]

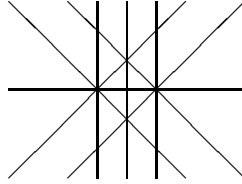


FIGURE 2. The  $B_3$ -arrangement

$$D(B_3) \simeq S(-1) \oplus S(-3) \oplus S(-5).$$

This configuration has 13 singular points, so if the singularities were in general position there would only be a two-dimensional space of quartics passing through the points. However, there are three quadruple points, and each set of lines through one of these points is a quartic vanishing on the singularities. A computation shows



that a generic quartic  $C$  in this three dimensional space is smooth, and that  $B_3 \cup C$  is quasihomogeneous.

By Proposition 3.3, the Hilbert polynomial and Hilbert function of the module  $\Gamma_* \mathcal{O}_C(-K_C - H - R)$  agree for  $t < \frac{21}{4}$  and  $t > \frac{29}{4}$ , and so applying Proposition 2.5 and Theorem 1.5 we have

$t$	4	5	6	7	8	9	10	11
$h^0((t-1)H - K_C - R)$	0	0	?	?	9	13	17	21

It remains to determine  $H^0((t-2)H - R)$  for  $t \in \{6, 7\}$ . The space  $H^0(4H - R)$  consists of quartics through the thirteen singular points of  $B_3$ . As observed above, this space has dimension 3, but  $C$  itself is one of the quartics, so  $h^0(4H - R) = 2$ . A direct calculation shows that  $h^0(5H - R) = 5$ , so

$$\mathrm{HS}(\Gamma_* \mathcal{O}_C(-K_C - H - R), t) = \frac{2t^6 + t^7 + t^8}{(1-t)^2} = \frac{2t^6 - t^7 - t^9}{(1-t)^3}.$$

Since  $\mathrm{Der}(\log B_3)(-4) \simeq S(-7) \oplus S(-9)$ , taking global sections in Theorem 1.5 and applying Proposition 1.6 shows  $\mathrm{Der}(\log B_3 \cup C)$  is free, with

$$\begin{aligned} \mathrm{HS}(\mathrm{Der}(\log B_3 \cup C), t) &= \mathrm{HS}(\mathrm{Der}(\log B_3)(-4), t) + \frac{2t^6 - t^7 - t^9}{(1-t)^3} \\ &= \frac{t^7 + t^9}{(1-t)^3} + \frac{2t^6 - t^7 - t^9}{(1-t)^3} \\ &= \frac{2t^6}{(1-t)^3}. \end{aligned}$$

Example 2.2 of [13] shows in general that  $\mathrm{sing}(\mathcal{A}) \neq \mathrm{sing}(\mathcal{A} \cup C)$ .

**Concluding remarks** Our work raises a number of questions:

- (1) Does this generalize to other surfaces? For the Hilbert polynomial arguments to work, the surface should possess an ample line bundle. More generally, does this generalize to higher dimensions? Note that [13] shows the quasihomogeneous property will be necessary.
- (2) The Hilbert series of  $\Gamma_* \mathcal{O}_C(-K_C - H - R)$  depends solely on a set of reduced points on a plane curve. If  $\mathcal{A} = \bigcup_{i=1}^r Y_i$  with  $Y_i$  reduced and irreducible, can an iterated construction using linkage yield the Hilbert series?
- (3) In [7], Liao gives a formula relating Chern classes of logarithmic vector fields to the Chern-Schwartz-MacPherson class of the complement, showing that on a surface the two are equal exactly when the singularities are quasihomogeneous, and in [1], Aluffi gives an explicit relation between the characteristic polynomial of an arrangement and the Segre class of the Jacobian scheme. Can one prove Theorem 1.5 using these methods?

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